Area Regge calculus and continuum limit

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Abstract

Encountered in the literature generalisations of general relativity to independent area variables are considered, the discrete (generalised Regge calculus) and continuum ones. The generalised Regge calculus can be either with purely area variables or, as we suggest, with area tensor-connection variables. Just for the latter, in particular, we prove that in analogy with corresponding statement in ordinary Regge calculus (by Feinberg, Friedberg, Lee and Ren), passing to the (appropriately defined) continuum limit yields the generalised continuum area tensor-connection general relativity.

The idea that Regge calculus should be formulated in terms of the areas of the triangles instead of the edge lengths [1] originates from the attempts to generalise the 3-dimensional Ponzano-Regge model of quantum gravity [2, 3] to the physical 4-dimensional case [4, 5]. One way of treating the area variables is by imposing certain geometrical constraints enforcing them to be still expressible in terms of the edge lengths [6]. Principially new theory arises, however, if one treats the areas as the fundamental and independent variables [7]. The basis for possibility and naturalness of such treatment in the physical 4-dimensional case is simple observation that the number 10 of the edges of a 4-simplex is the same as the number of it's triangular faces. Therefore the both sets, those of area and length variables turn out to be expressible in terms of each other inside a given 4-simplex, at least locally. The requirement for the neighbouring 4-simplices to have coinciding lengths of (up to 6) common edges is relaxed to the requirement of having only coinciding areas of (up to 4) common triangles instead. This leaves the lengths ambiguous in general, but still allows to define the dihedral angles in each the 4-simplex. Therefore the angle defects φ_Δ on the triangles Δ are defined as functions of the triangle areas A_Δ, and Regge calculus action still can be written out,

$$S_{\rm area} = \sum_\Delta A_\Delta \varphi_\Delta(\{A_\Delta\}). \tag{1}$$

Variation w.r.t. A_Δ is performed as in the ordinary Regge calculus, but since variations δA_Δ for the different Δ are now independent values, the equations of motion give $\varphi_\Delta = 0$ [7]. Because of the lack of metric and of the usual geometric interpretation of φ_Δ, this does not mean flat spacetime [7]. Moreover, it was mentioned [8] that diagonalising second variation of the action (1) as bilinear form of δA_Δ around a (distorted) hypercubic lattice results in the same dynamical content (the number of the dynamical degrees of freedom) as in the usual Regge calculus based on the edge length variables [9].

An interesting question is that of possible continuum counterpart of the area Regge calculus with the action (1). On the other hand, the Hilbert-Palatini form of the usual general relativity admits generalisation to area variables in a natural way. Indeed, write the action in the form

$$S_{\rm HP} = \frac{1}{4}\int d^4x\, \pi^{\lambda\mu}_{ab}[\mathcal{D}_\lambda, \mathcal{D}_\mu]^{ab} \tag{2}$$

where $\mathcal{D}_\lambda = \partial_\lambda + \omega_\lambda$ (in fundamental representation) is covariant derivative, and $\omega^{ab}_\mu = -\omega^{ba}_\mu$ is element of $so(3,1)$, Lie algebra of $SO(3,1)$ group in the Lorentzian case or an element of $so(4)$, Lie algebra of $SO(4)$ in the Euclidean case. λ, μ, $\ldots = 1, 2, 3, 4$ are coordinate indices and a, b, $\ldots = 1, 2, 3, 4$ are local ones. The antisymmetric in a, b and in λ, μ *area tensor* $\pi^{\lambda\mu}_{ab}$ is subject to the tensor relation

$$\pi^{\lambda\mu}_{ab}\pi^{\nu\rho}_{cd}\epsilon^{abcd} \sim \epsilon^{\lambda\mu\nu\rho}. \tag{3}$$

This equation simply ensures that a tetrad e^a_λ exists so that $\pi^{\lambda\mu}_{ab}$ is a bivector,

$$\pi^{\lambda\mu}_{ab} = \frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\epsilon_{abcd}e^c_\nu e^d_\rho. \tag{4}$$

More often treated as area tensor is the twice dual to $\pi^{\lambda\mu}_{ab}$,

$$v^{ab}_{\lambda\mu} = \frac{1}{4}\epsilon^{abcd}\epsilon_{\lambda\mu\nu\rho}\pi^{\nu\rho}_{cd}, \tag{5}$$

which in the tetrad formalism reduces to $e^a_\lambda e^b_\mu$ - $e^b_\lambda e^a_\mu$. Generalisation simply amounts to omitting the eq. (3) so that the components of area tensor become independent variables. In fact, this generalisation is the subject of study in the literature when Ashtekar formalism [10] is discussed. Indeed, Ashtekar formalism can be obtained by separating self- and antiselfdual (over local indices) parts of ω^{ab}_λ, $\pi^{\lambda\mu}_{ab}$ in eqs. (2, (3), the eq. (3) being an issue point for the *reality conditions* (in the case of the Lorentzian signature) in this formalism [11, 12]. The hope is that this (in fact, more simple) formalism with unrestricted area tensor can be solved and the reality conditions can be imposed anyhow afterwards to select a real section of the complex phase space [13]. An yet unresolved problem encountered in this way is how do classical configurations of the gravitational field like gravitons arise. In this respect, there is the difference from the area Regge calculus based on the action (1) where the dynamical degrees of freedom probably match those in the ordinary length Regge calculus as mentioned above.

Thus, we have the two generalisations of general relativity to independent area variables, the discrete (1) and continuum (2) ones which use the purely area and area tensor - connection variables, respectively. What is the connection between these two? When linking the ordinary length (metric) and tetrad - connection formalisms one uses the eq. $g_{\lambda\mu} = e^a_\lambda e^a_\mu$ relating the different sets of independent variables; then it is noted that the metric tensor $g_{\lambda\mu}$ has the same number of components as the number of edges of a 4-simplex. Now in analogy one could like to construct the "area metric tensor" $h_{\lambda\mu\nu\rho} = v^{ab}_{\lambda\mu}v^{ab}_{\nu\rho}$ which being viewed as a 6×6 symmetrical matrix w.r.t. antisymmetrical pairs $[\lambda\mu]$ and $[\nu\rho]$ has 21 independent components, not the same as the number 10 of independent areas of a 4-simplex. Therefore we cannot say that there is a natural equivalence between the both considered area variable formalisms. At least, either the 4-simplex cannot serve as an elementary cell of the corresponding "area geometry" (defined by $h_{\lambda\mu\nu\rho}$) or the field $h_{\lambda\mu\nu\rho}$ is subject to some additional constraints. Indirectly, inequivalence between the two formalisms displays also in the above mentioned different dynamical content of them.

On the other hand, the tetrad-connection (including the case of self-dual connection) representation of ordinary Regge calculus has been suggested by the author [14],

$$S(V, \Omega) = \sum_{\sigma^2} |V_{\sigma^2}| \arcsin \frac{V_{\sigma^2} * R(\Omega)}{|V_{\sigma^2}|} \tag{6}$$

where $V^{ab}_{\sigma^2}$ are the bivectors of the 2-faces σ^2, $|V|^2 \equiv \frac{1}{2}V^{ab}V^{ab}$, $R_{\sigma^2}(\Omega)$ is the product of the SO(4) in the Euclidean (SO(3,1) in the Lorentzian) case matrices Ω_{σ^3} living on the 3-faces σ^3 taken along the loop enclosing the given 2-face σ^2, $V * R \equiv \frac{1}{4}V^{ab}R^{cd}\epsilon_{abcd}$. Strictly speaking, the bivector carries one else subscript σ^4 as V_{σ^2,σ^4} indicating the local frame where the bivector of a given 2-face σ^2 is defined. Also the eq. (6) should be

accomplished with geometrical constraints ensuring the bivector form of $V^{ab}_{\sigma^2}$. The form of these constraints can be taken very simple, linear and bilinear, although at the price of rather large number of them, if we extend the set of $V^{ab}_{\sigma^2,\sigma^4}$ to all $\sigma^4 \supset \sigma^2$ [15]. Generalisation to independent area tensor formalism is by simply omitting these constraints, and we are left with eq. (6) alone with freely varied V_{σ^2} and Ω_{σ^3} variables.

Write out the table of actions for possible versions of area general relativity.

variables / discreteness	area	area tensor -connection
continuum	?	$\int \pi(\partial\omega + \omega \wedge \omega)$
discrete	$\sum A\varphi(A)$	$\sum \|V\| \arcsin \frac{V*R}{\|V\|}$

An interesting problem besides that of filling in the upper left cell of the table is that of establishing correspondence between the different cells. In the present paper we obtain the area tensor-connection continuum action (2) from the discrete one (6) in the (properly defined) continuum limit (an analog of the theorem by Feinberg, Friedberg, Lee and Ren [16] for the ordinary edge length Regge calculus).

First choose Regge lattice of a certain periodic structure used in [9]. Topologically, the Regge manifold periodic cell is a 4-cube divided into 24 4-simplices sharing the hyperbody diagonal. Let the indices λ, μ, $\ldots = 1, 2, 3, 4$ label the cube edges emerging from a vertex O along the corresponding coordinate axes. The T_λ, $T^{-1}_\lambda = \overline{T}_\lambda$ are operators of the translations to the two neighbouring vertices in the positive and negative directions of λ. Introduce multiindices A, B, C, $\ldots$, the unordered sequences of different indices, e. g. $A = (\lambda\mu\ldots\nu)$. The link (1-simplex) connecting the points O and $T_\lambda T_\mu \ldots T_\nu O$ will be labelled just by A while the k-simplex at $k > 1$ spanned by the links A_1, (A_1A_2), $\ldots$, $(A_1A_2\ldots A_k)$ will be denoted by the ordered sequence of multiindices $[A_1A_2\ldots A_k]$. Here the symbol $(A_1A_2\ldots A_i)$ means multiindex composed of all the indices encountered in A_1, A_2, $\ldots$, A_i. If there can be no confusion, the round and square brackets will be omitted: notation '$[AB]$' is equivalent to 'the 2-simplex AB' etc. On the whole, we have the following simplices attributed to the given vertex O:
(i) 15 links λ, $\lambda\mu$, $\lambda\mu\nu$, 1234;
(ii) 50 2-simplices (triangles) $\lambda\mu$, $(\lambda\mu)\nu$, $\lambda(\mu\nu)$, $\lambda(\mu\nu\rho)$, $(\lambda\mu)(\nu\rho)$, $(\lambda\mu\nu)\rho$;
(iii) 60 3-simplices $\lambda\mu\nu$, $(\lambda\mu)\nu\rho$, $\lambda(\mu\nu)\rho$, $\lambda\mu(\nu\rho)$ (the latter three symbols will be also more briefly written as d$\nu\rho$, λdρ, $\lambda\mu$d, respectively, the "d" meaning "diagonal");
(iv) 24 4-simplices $\lambda\mu\nu\rho$.

To each oriented 3-simplex σ^3 shared by the 4-simplices σ^4_1 and σ^4_2 we assign the SO(4) (SO(3,1)) matrix $\Omega^{\epsilon(\sigma^4_1,\sigma^4_2)}_{\sigma^3}$ in the Euclidean (Lorentzian) case which acts from the local frame of σ^4_1 to that of σ^4_2; the choice of $\epsilon(\sigma^4_1, \sigma^4_2) = \pm 1$ just specifies orientation of

$\sigma^3 = \sigma^4_1 \cap \sigma^4_2$. To each 2-simplex σ^2 we assign a simplex $\sigma^4 \supset \sigma^2$ in the frame of which area tensor V_{σ^2} is defined; then R_{σ^2}, the product of matrices $\Omega^{\pm 1}_{\sigma^3}$ acting along the loop enclosing σ^2 acts from this σ^4 to itself. Our choice of orientation of the 3-simplices and of the frames of definition of area tensors corresponds to the following expressions for the curvature matrices,

$$
\begin{aligned}
R_{41} &= \overline{\Omega}_{413}(\overline{T}_2\overline{\Omega}_{241})(\overline{T}_{23}\overline{\Omega}_{\mathrm{d}41})(\overline{T}_3\Omega_{341})\Omega_{412}\Omega_{41\mathrm{d}},\\
R_{4(23)} &= \overline{\Omega}_{4\mathrm{d}1}\overline{\Omega}_{423}(\overline{T}_1\Omega_{14\mathrm{d}})\Omega_{432},\\
R_{23} &= \overline{\Omega}_{23\mathrm{d}}\overline{\Omega}_{231}(\overline{T}_4\overline{\Omega}_{423})(\overline{T}_{14}\Omega_{\mathrm{d}23})(\overline{T}_1\Omega_{123})\Omega_{234},\\
R_{2(43)} &= \overline{\Omega}_{2\mathrm{d}1}\overline{\Omega}_{234}(\overline{T}_1\Omega_{12\mathrm{d}})\Omega_{243},\\
R_{(24)3} &= \overline{\Omega}_{\mathrm{d}31}\overline{\Omega}_{243}(\overline{T}_1\Omega_{1\mathrm{d}3})\Omega_{423},\\
R_{1(32)} &= \overline{\Omega}_{132}(\overline{T}_4\overline{\Omega}_{41\mathrm{d}})\Omega_{123}\Omega_{1\mathrm{d}4},\\
R_{1(432)} &= \overline{\Omega}_{1\mathrm{d}4}\Omega_{12\mathrm{d}}\Omega_{1\mathrm{d}3}\Omega_{14\mathrm{d}}\overline{\Omega}_{1\mathrm{d}2}\overline{\Omega}_{13\mathrm{d}},\\
R_{(14)(32)} &= \overline{\Omega}_{14\mathrm{d}}\Omega_{\mathrm{d}23}\Omega_{41\mathrm{d}}\overline{\Omega}_{\mathrm{d}32},\\
&\ldots\ \mathrm{cycle}(1,2,3)\ \ldots,\\
R_{4(123)} &= \Omega_{4\mathrm{d}3}\overline{\Omega}_{42\mathrm{d}}\Omega_{4\mathrm{d}1}\overline{\Omega}_{43\mathrm{d}}\Omega_{4\mathrm{d}2}\overline{\Omega}_{41\mathrm{d}}.
\end{aligned}
\tag{7}
$$

These eqs. define 25 expressions. The remaining half of the whole number 50 of curvature matrices can be obtained by permuting groups of indices: if $R_{(\lambda\ldots\mu)(\nu\ldots\rho)} = \prod T^{\pm 1}_{(\ldots)}\ \Omega^{\pm 1}_{\ldots\lambda\ldots\mu\nu\ldots\rho\ldots}$ then $\overline{R}_{(\nu\ldots\rho)(\lambda\ldots\mu)} = \prod T^{\pm 1}_{(\ldots)}\ \Omega^{\pm 1}_{\ldots\nu\ldots\rho\lambda\ldots\mu\ldots}$. This completely defines the tensor area-connection Regge-type action $S(V,\Omega)$.

The variables Ω_{σ^3}, V_{σ^2} are in the natural way the functions of the vertices O to which the given σ^3, σ^2 are attributed. To pass to the continuum limit we suppose that these variables are the particular values of some smooth functions $\Omega(x)$, $V(x)$ on the spacetime continuum taken at the locations of the vertices. Then we uniformly tend the coordinate differences between the neighbouring vertices to zero (thus enlarging the number of vertices in any finite region). The analog of the derivative, $T_\lambda - 1$ should be of the order of ε, the typical coordinate difference between the neighbouring vertices. For the continuum limit leading to an expression of the type of S_{HP} be defined it turns out natural to ascribe the following orders of magnitude in $\varepsilon \to 0$ to the discrete values in question,

$$V = O(\varepsilon^2),\ w = O(\varepsilon)\ (\exp w \equiv \Omega), \tag{8}$$

$$V_{BA} = -V_{AB} + O(\varepsilon^3), \tag{9}$$

$$w_{ABC} = w_{BAC} + O(\varepsilon^2) = w_{ACB} + O(\varepsilon^2), \tag{10}$$

which correspond to the naive considerations that we deal with tensors V of the closely located almost parallel (up to $O(\varepsilon)$) 2-simplices AB, BA and matrices w for the parallel vector transport in the almost parallel directions orthogonal to the 3-simplices ABC, BAC, ACB at almost equal (up to $O(\varepsilon^2)$) distances $O(\varepsilon)$ separating centers of almost similar 4-simplices. Of course, a geometric interpretation is valid only for the particular case when area variables correspond to certain edge length (tetrad) ones, but the orders

of magnitude presented turn out to be justified in what follows from the purely formal computational grounds as well.

In the continuum limit the area tensors $V_{\lambda\mu}$ directly correspond to $v_{\lambda\mu}$ in $S_{\rm HP}$, eqs. (2), (5) (more exactly, to $\varepsilon^2 v_{\lambda\mu}$). In order to reduce area tensors of other 2-simplices to $v_{\lambda\mu}$ we need relations of the type $V_{\lambda(\mu\nu)} = V_{\lambda\mu} + V_{\lambda\nu}$. The latter in the usual tetrad formalism would follow (in the leading order in ε when one can neglect rotations needed to express different tensors in the same frame) from representation of the link vector $l^a_{\mu\nu}$ as the sum of l^a_μ and l^a_ν. What consequence of the theory could, in principle, provide us with relations of such type is only the Gauss law accessible under some assumptions via eqs. of motion for Ω from $S(V,\Omega)$. That is, the limiting expression of the type of $S_{\rm HP}$ can be obtained only upon partial use of the eqs. of motion. For that we take first and second orders in the expansion of $S(V,\Omega)$ in w,

$$S = S^{(1)}(V,w) + S^{(2)}(V,w,w) \tag{11}$$

where $S^{(1)}$, $S^{(2)}$ are first and second order forms in w, both linear in V. According to eq. (8) higher orders in w give vanishing at $\varepsilon \to 0$ contribution as a sum of the terms $O(\varepsilon^5)$ over cells number of which in a fixed finite region is $O(\varepsilon^{-4})$. Partially using eqs. of motion for w reduces this equation to

$$- S = +S^{(2)}(V,w,w). \tag{12}$$

Here it is sufficient to know V in the leading order in ε. Write out the eq. of motion for Ω which plays the role of the Gauss law. For any σ^2 and $\sigma^3 \supset \sigma^2$ we have $R_{\sigma^2} = (\Gamma_1\Omega_{\sigma^3}\Gamma_2)^\epsilon$ where Γ_1, $\Gamma_2 \in$ SO(4) (SO(3,1)) and $\epsilon = \pm 1$ are functions of σ^2, σ^3. Then for a given σ^3

$$\begin{aligned}\sum_{\sigma^2\subset\sigma^3} \epsilon(\sigma^2,\sigma^3)\Gamma_2(\sigma^2,\sigma^3) \Big[& V(\sigma^2){\rm tr}R(\sigma^2) - V(\sigma^2)R(\sigma^2)^{-\epsilon(\sigma^2,\sigma^3)} \\ & - R(\sigma^2)^{\epsilon(\sigma^2,\sigma^3)}V(\sigma^2)\Big]\overline{\Gamma}_2(\sigma^2,\sigma^3)\frac{1}{\cos\varphi(\sigma^2)} = 0, \end{aligned} \tag{13}$$

$$\sin\varphi(\sigma^2) = \frac{V(\sigma^2) * R(\sigma^2)}{|V(\sigma^2)|}$$

[14]. With taking into account the eq. (10) we have $R(\sigma^2) = \mathbf{1} + O(\varepsilon^2)$ (connection matrices enter the products defining curvature matrices as pairs Ω_1, $\overline{\Omega}_2$ where Ω_1 and Ω_2 are approximately equal up to $\mathbf{1} + O(\varepsilon)$). Therefore in the leading and next-to-leading orders in ε we have the Gauss law

$$\sum_{\sigma^2\subset\sigma^3} \epsilon(\sigma^2,\sigma^3)\Gamma_2(\sigma^2,\sigma^3)V(\sigma^2)\overline{\Gamma}_2(\sigma^2,\sigma^3) = O(\varepsilon^4). \tag{14}$$

On our particular Regge lattice and with taking into account (9) the leading order reads

$$V_{\lambda(\mu\nu)} + V_{\nu(\lambda\mu)} + V_{\mu\lambda} + V_{\mu\nu} = O(\varepsilon^3) \tag{15}$$

and

$$V_{(\lambda\mu)(\nu\rho)} + V_{\rho(\lambda\mu\nu)} + V_{\nu(\lambda\mu)} + V_{\nu\rho} = O(\varepsilon^3). \tag{16}$$

Introduce the quantities δ_{AB} which correct the naive (inspired by the tetrad formalism) decompositions of the tensors V_{AB} in terms of those tensors of the simplest triangles $\lambda\mu$; for example,

$$V_{\lambda(\mu\nu)} = V_{\lambda\mu} + V_{\lambda\nu} + \delta_{\lambda(\mu\nu)}. \tag{17}$$

Then

$$\begin{aligned} \delta_{\lambda(\mu\nu)} + \delta_{\nu(\lambda\mu)} &= O(\varepsilon^3), & (18)\\ \delta_{(\lambda\mu)(\nu\rho)} + \delta_{\rho(\lambda\mu\nu)} + \delta_{\nu(\lambda\mu)} &= O(\varepsilon^3) & (19) \end{aligned}$$

from the eqs. (15) and (16), respectively. The approximate antisymmetry of V_{AB} w. r. t. A, B (eq. (9)) also holds for δ_{AB}. The eq. (18) means that $\delta_{\lambda(\mu\nu)}$ (symmetrical w. r. t. the second and third indices by definition) is approximately antisymmetrical in the first and second (third) indices. It is easy to see that such the quantity can be equal only to zero with the same accuracy. Substituting this result into the eq. (19) we see that $\delta_{(\lambda\mu)(\nu\rho)}$ is approximately symmetrical in λ, ν and, consequently, in μ, ρ and, therefore, in the multiindices $(\lambda\mu)$, $(\nu\rho)$. Together with the above stated antisymmetry in multiindices this means that the related δ's vanish up to $O(\varepsilon^3)$ as well. Thus, the naive expressions expectable from the analogy with projecting area bivectors onto the coordinate planes in the usual tetrad formalism hold in our case too,

$$\begin{aligned} V_{\lambda(\mu\nu)} &= V_{\lambda\mu} + V_{\lambda\nu} + O(\varepsilon^3), & (20)\\ V_{\lambda(\mu\nu\rho)} &= V_{\lambda\mu} + V_{\lambda\nu} + V_{\lambda\rho} + O(\varepsilon^3), & (21)\\ V_{(\lambda\mu)(\nu\rho)} &= V_{\lambda\nu} + V_{\lambda\rho} + V_{\mu\nu} + V_{\mu\rho} + O(\varepsilon^3). & (22) \end{aligned}$$

Remarkably is that these more stringent than the Gauss law relations turn out to be the consequences of it in some assumptions (that the variables vary slowly between neighbouring simplices).

Now it is straightforward to substitute the decompositions of area tensors V_{AB} over "elementary" ones $V_{\lambda\mu}$ obtained into the second order in w part of the action, eq. (12). The latter, in turn, is contributed by the bilinear in w antisymmetric parts of the curvature matrices $R^{(2)}_{AB}$. For example,

$$R^{(2)}_{23} = [w_{23\mathrm{d}}, w_{231}] + [w_{23\mathrm{d}}, w_{423}] + [w_{231}, w_{423}]. \tag{23}$$

Here translation operators T_λ are substituted by the unity with the accuracy of $O(\varepsilon^3)$. The proportional to V_{23} bilinear in w parts of the action are contained in $V_{AB} * R^{(2)}_{AB}$ at $AB = 23$, $(21)3$, $(24)3$, $2(31)$, $2(34)$, $(214)3$, $2(143)$, $(21)(34)$, $(24)(13)$. Collecting these contributions we find

$$\sum_{AB} V_{AB} * R^{(2)}_{AB} = V_{23} * [w_1, w_4] + \ldots = \frac{1}{4} \sum_{\lambda\mu\nu\rho} V_{\lambda\mu} * [w_\nu, w_\rho]\epsilon_{\lambda\mu\nu\rho} \tag{24}$$

where

$$
\begin{aligned}
w_1 &= w_{234} - w_{34\mathrm{d}} + w_{24\mathrm{d}} - w_{23\mathrm{d}}, \\
&\ldots \ \mathrm{cycle}(1,2,3) \ \ldots, \\
-w_4 &= w_{123} + w_{12\mathrm{d}} + w_{23\mathrm{d}} + w_{13\mathrm{d}}.
\end{aligned} \tag{25}
$$

Thus the continuum action as bilinear form of ω (following upon partial use of the eqs. of motion) is obtained from the corresponding discrete version (12) in the continuum limit, by identifying $V_{\lambda\mu}$ with $\varepsilon^2 v_{\lambda\mu}$ and w_λ with $\varepsilon\omega_\lambda$. To reproduce the action in the standard form with independent ω we need the equations of motion which should be derived themselves from the discrete version and then used in backward direction as compared to how we have obtained the discrete action as bilinear in w, eq. (12) from the original linear plus bilinear expression with independent w, eq. (11). The equations of motion are just the Gauss law which has been already written out, eq. (14), but now the linear in w part is of interest, i. e. it should be expanded to the next-to-leading order in ε. The Gauss law in the continuum theory expresses closure of the surface of infinitesimal 3-cube. Consider the 3-cube of the Regge lattice laying, say, in the 123-hyperplane and composed of six 3-simplices (123 and permutations) and write out the eq. (14) for each of these simplices, e. g.

$$
-\Omega_{12\mathrm{d}} V_{12} \overline{\Omega}_{12\mathrm{d}} + \Omega_{1\mathrm{d}4} V_{1(32)} \overline{\Omega}_{1\mathrm{d}4} + T_1(\Omega_{234} V_{23} \overline{\Omega}_{234}) - \Omega_{\mathrm{d}34} V_{(21)3} \overline{\Omega}_{\mathrm{d}34} = O(\varepsilon^4) \tag{26}
$$

for $\sigma^3 = 123$ and

$$
\Omega_{13\mathrm{d}} V_{13} \overline{\Omega}_{13\mathrm{d}} - V_{1(32)} + V_{(13)2} - T_1(\Omega_{324} V_{32} \overline{\Omega}_{324}) = O(\varepsilon^4) \tag{27}
$$

for $\sigma^3 = 132$. Excluding $V_{1(32)}$ from these equations we get to the linear order in w

$$
\begin{aligned}
&- V_{12} + V_{13} + V_{(13)2} - V_{(21)3} + T_1(V_{23} - V_{32}) - [w_{12\mathrm{d}}, V_{12}] + [w_{1\mathrm{d}4} + w_{13\mathrm{d}}, V_{13}] \\
&\qquad + [w_{1\mathrm{d}4}, V_{(13)2}] - [w_{1\mathrm{d}4} + w_{324}, V_{32}] + [w_{234}, V_{23}] - [w_{\mathrm{d}34}, V_{(21)3}] = O(\varepsilon^4).
\end{aligned} \tag{28}
$$

This can be summed up with two other cyclic permutations of 1, 2, 3. In the terms $[w, V]$ the leading in ε accuracy of the definition of V is sufficient, and these area tensors can be reduced to $V_{\lambda\mu}$. In the terms of zero order in w the area tensors other than $V_{\lambda\mu}$ are cancelled in the overall sum. Thus we obtain

$$
(T_1 - 1)V_{23} + [w_1, V_{23}] + \mathrm{cycle}(1,2,3) = O(\varepsilon^4). \tag{29}
$$

This just reduces to the $\rho = 4$ component of the continuum Gauss law,

$$
(\partial_\lambda v_{\mu\nu} + [\omega_\lambda, v_{\mu\nu}])\epsilon^{\lambda\mu\nu\rho} = 0. \tag{30}
$$

This allows to rewrite eq. (24) (upon replacing V, w by $\varepsilon^2 v$, $\varepsilon\omega$ in the continuum limit) in the standard linear plus bilinear form which reproduces the continuum action S_{HP}.

Thus, the continuum action (2) can be obtained from the discrete Regge-type one (6) under quite reasonable assumptions (eqs. (8), (9), (10)) defining the continuum limit,

and with partial use of the equations of motion. Since the most general case when area tensors are independent variables was considered, the same is valid also for the tetrad (bivector) form of these variables, i. e. for the usual general relativity and Regge calculus in the tetrad-connection form. Besides that, we can restrict the connection matrices Ω in the action (6) to have the (anti-)selfdual generators, and this still presents the Regge calculus action [14]. Evidently, the continuum limit action in this case will be given just by eq. (2) where now ω^{ab}_{λ} and therefore $V^{ab}_{\lambda\mu}$ are restricted to be (anti-)selfdual matrices.

The remaining problem concerning the interrelation between the continuum and discrete area-connection theories is that of validity of an analog of the Friedberg-Lee theorem [17]. That is, whether discrete Regge-type action can be obtained by the exact calculation of the continuum action on a particular distribution of the ω, π fields, a kind of conical singularities? The positive answer would mean that area-connection generalisation of Regge calculus be the second example (after usual Regge calculus) of the discrete *minisuperspace* theory which at the same time is able to approximate the continuum counterpart with arbitrarily large accuracy.

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